Lec 20: 10/12/2009

Path Integral Formulation of Quantum Mechanics:

The formulation of quantum mechanics we have considered so far is due to Schrodinger. In this formulation the classical conjugate variable x, p are promoted to operators with the commutation relation $[X, P]$ s.t.

Then the Hamiltonian H will become an operator:

$$H = \frac{p^2}{2m} + V(x)$$

The relevant equation governing the dynamics is the Schrodinger equation:

$$H|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle$$

The propagator $U(t)$ is given by:

$$U(t) = e^{-\frac{iHt}{\hbar}} \quad \text{(assuming that } \frac{\partial H}{\partial t} = 0\text{)}$$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

The matrix elements of $U(t)$ in the position basis

are:

$$\langle x | U(t) | x' \rangle \equiv U(x,t;x',0) = \sum_n e^{-i\frac{E_n t}{\hbar}} \psi_n^*(x') \psi_n(x)$$

This formulation is based on the Hamiltonian formalism of mechanics. It is called "canonical quantization" as well.

There exists another formulation of quantum mechanics according to Feynman. It is based on the Lagrangian formalism of mechanics. In this formulation, called "Path integral quantization", the matrix element of the propagator is given by:

$$U(x,t;x',0) = A \sum_{\substack{\text{all} \\ \text{paths}}} e^{\frac{-iS}{\hbar}}$$

Here S represents the action for a path that connects points (x,t) and $(x',0)$. The sum is over all paths, and A is an overall normalization factor.

To elaborate, lets first recall how the Lagrangian formalism work in the classical mechanics.

Lagrangian Mechanics (Classical Level):

$$L = T - V = \frac{1}{2} m\dot{q}^2 - V_{(q)}$$

According to the Newton's second law, the trajectory of the particle is given by the solution to the following differential equation.

$$m\ddot{q} = -V'_{(q)} \qquad q_{(\cdot)}, \dot{q}_{(\cdot)} \text{ initial Conditions}$$

Lagrangian formulation states that the trajectory of the particle between points $(q', 0)$ and (q, t) is the path with the least action.

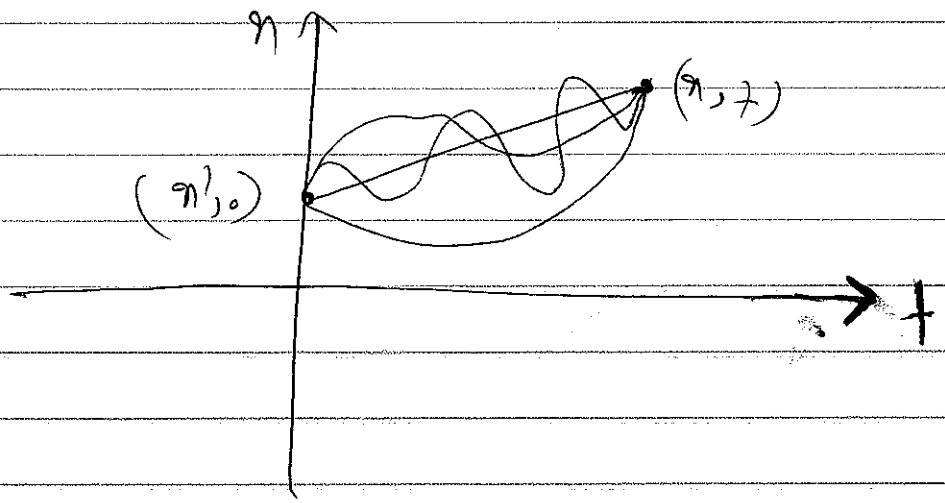
$$S = \int_0^t L \, dt = \int_0^t \left[\frac{1}{2} m\dot{q}^2 - V_{(q)}\right] dt$$

It can be shown that for the path with the least action, the Euler-Lagrange equation is satisfied.

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \Rightarrow m\ddot{q} + V'_{(q)} = 0$$

The right-hand side is nothing but the equation of motion from Newton's second law.

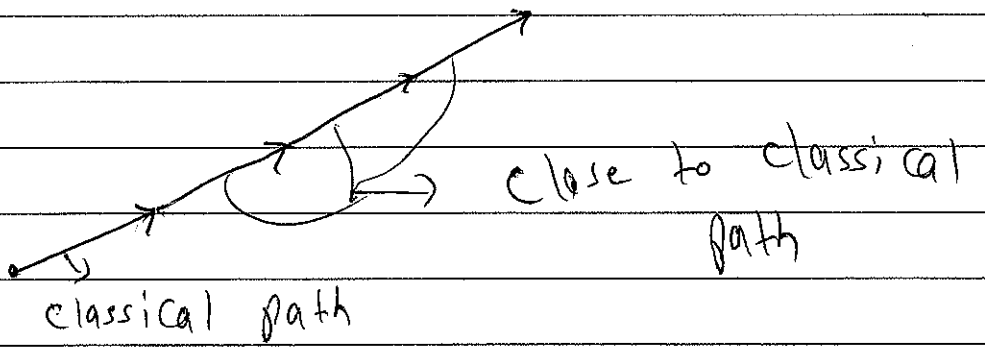
In the path integral quantization, the particle can go along any path between points $(q', 0)$ and (q, t). One should compute S along each path, and then sum all factors $e^{-iS/\hbar}$ up.



Performing the summation $\sum_{\text{all paths}} e^{-iS/\hbar}$ is the main task to do. We note that these are just phases since S is a real number. Therefore, this amounts to adding unit vectors with different phases.

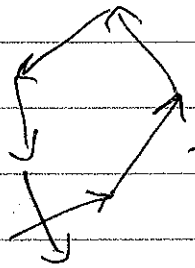
The classical path has the least action. Describing the action S as a function of the path, S has a minimum at the classical path. Thus, the action S changes slowly for paths that are sufficiently close to the classical trajectory (we will discuss what close means later on).

As a result, for paths that are close enough to the classical path, the phase change slowly, and the corresponding unit vectors line up in parallel:



On the other hand, for paths far from the classical trajectory, the action changes quickly as a function of path. Hence, the phase of the corresponding

unit vector also change quickly and can easily go over a 2π change as the paths change.



→ far from classical path

The main contribution to the sum $\sum_{\text{all paths}} e^{-iS/\hbar}$ therefore comes from the paths that are sufficiently close to the classical trajectory.

It turns out that for potentials up to quadratic in position $V = a + bx + cx^2$, all that matters is the classical path. For these potentials we have (proof in the book):

$$\sum_{\text{all paths}} e^{-iS/\hbar} = A' e^{\frac{-iS_{cl}}{\hbar}}$$

Here A' is just a constant, eventually absorbed in A.

For this type of potentials (including free particle and harmonic oscillator), a great deal of information can be obtained by computing S_{cl}. In this case:

$$U(x,t;x',0) = A\, e^{\frac{-iS_{cl}}{\hbar}}$$

A is a constant that depends on time in general. Even without calculating A, we can find all the relative probabilities.

In the case of free particle S_{cl} can be computed easily. This is much simpler than calculating $U(x,t;x',0)$ from Schrodinger formulation where we had to evaluate Gaussian integrals. The difference is even bigger in the case of harmonic oscillator, where $\sum_n \psi_n^*(x')\,\psi_n(x)\, e^{-\frac{iE_n t}{\hbar}}$ is not easy to find. It is much simpler to compute S_{cl} instead.